

Brownian motion with singular drift: small time asymptotics and associated boundary value problems

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Consider a Brownian motion with singular drift on R^d with $d \geq 3$ whose infinitesimal generator is $\frac{1}{2}\Delta + \mu \cdot \nabla$, where each μ_i of $\mu = (\mu_1, \dots, \mu_d)$ is a measure in some suitable Kato class. The talk has two parts.

Part I: Small time asymptotics for the Brownian motion with singular drift (with Zhen-Qing Chen, Shizan Fang).

Part II: Associated Dirichlet boundary value problems (with Saisai Yang).

Part I: Introduction and framework

We are interested in the pointwise small time asymptotic property of the heat kernel of Brownian motion with singular drifts in R^d with $d \geq 3$; that is,

$$dX_t = dW_t + dA_t \quad \text{with } X_0 = x, \quad (0.1)$$

where A is a continuous additive functional of X having “Revuz measure μ ”. Informally $\{X_t, t \geq 0\}$ is a diffusion process in R^d with generator $\frac{1}{2}\Delta + \mu \cdot \nabla$, where $\mu = (\mu_1, \dots, \mu_d)$ is a vector-valued signed measure on R^d belonging to the Kato class $K_{d,1}$ to be introduced below. When $\mu_j(dx) = b_j(x)dx$ for some function b_j , X is a solution to the stochastic differential equation

$$dX_t = dW_t + b(X_t)dt \quad \text{with } X_0 = x \quad (0.2)$$

Definition 1.1. A signed measure ν on R^d with $d \geq 3$ is said to be in the Kato class $K_{d,k}$ (for $k = 1, 2$) if

$$\lim_{r \downarrow 0} \sup_{x \in R^d} \int_{|x-y| \leq r} \frac{|\nu|(dy)}{|x-y|^{d-k}} = 0,$$

where $|\nu|$ is the total variation of ν . A measurable function f on R^d is said to be in the Kato class $K_{d,k}$ (for $k = 1, 2$) if $|f(x)|dx \in K_{d,k}$.

Clearly any bounded measurable functions are in the Kato class $K_{d,k}$ for $k = 1, 2$. By Hölder inequality, it is easy to see that $L^p(R^d; dx) \subset K_{d,1}$ for $p > d$ and $L^p(R^d; dx) \subset K_{d,2}$ for $p > d/2$. But a measure in $K_{d,1}$ can be quite singular. It is known that for a Borel measure μ on R^d , if there are constants $\kappa > 0$ and $\gamma > 0$ so that $\mu(B(x, r)) \leq \kappa r^{d-1+\gamma}$ for all $x \in R^d$ and $r \in (0, 1]$, then $\mu \in K_{d,1}$. Thus in particular, if $A \subset R^d$ is an Alfvors λ -regular set with $\lambda \in (d-1, d]$, then the Hausdorff measure \mathcal{H}^λ restricted to A is in $K_{d,1}$.

Part I: Introduction and framework

To recall the precise definition of a Brownian motion with a measure drift $\mu = (\mu_1, \dots, \mu_d)$ with $\mu_i \in K_{d,1}$ for $1 \leq i \leq d$, fix a non-negative smooth function φ in R^d with compact support and $\int \varphi(x) dx = 1$. For any positive integer n , we put $\varphi_n(x) = 2^{nd} \varphi(2^n x)$. For $1 \leq i \leq d$, define

$$b_i^{(n)}(x) = \int \varphi_n(x - y) \mu_i(dy)$$

Put $b^{(n)} = (b_1^{(n)}, \dots, b_d^{(n)})$. The following definition is taken from [BC].

Definition 1.2. A Brownian motion with drift μ is a family of probability measures $\{P_x, x \in R^d\}$ on $C([0, \infty), R^d)$, the space of continuous functions on $[0, \infty)$, such that under each P_x the coordinator process X has the decomposition

$$X_t = x + W_t + A_t$$

where

Part I: Introduction and framework

(a) $A_t = (A_t^{(1)} \dots, A_t^{(d)}) = \lim_{n \rightarrow \infty} \int_0^t b^{(n)}(X_s) ds$ uniformly in t over finite intervals, where the convergence is in probability;

(b) there exists a subsequence $\{n_k\}$ such that for every $t > 0$,

$$\sup_{k \geq 1} \int_0^t |b^{n_k}(X_s)| ds < \infty \quad a.s.$$

(c) W_t is a standard Brownian motion in R^d starting from the origin.

It is established in [BC] that, when each μ_i is in the Kato class $K_{d,1}$, Brownian motion with drift $\mu = (\mu_1, \dots, \mu_d)$, denoted by $\{X_t, t \geq 0\}$, exists and is unique in law for every starting point $x \in R^d$. Moreover, it is shown there that $X = \{X_t, t \geq 0; P_x, x \in R^d\}$ forms a conservative Feller process on R^d .

Later, it is obtained in [KS] that X has a jointly continuous transition density function $q(t, x, y)$ which admits the following Gaussian type estimate:

$$C_1 e^{-C_2 t} t^{-d/2} e^{-C_3 |x-y|^2/t} \leq q(t, x, y) \leq C_4 e^{C_5 t} t^{-d/2} e^{-C_6 |x-y|^2/t} \quad (0.3)$$

for all $(t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$, where $C_i, 1 \leq i \leq 6$, are some positive constants.

In this part, we are concerned with the precise Varadhan type small time asymptotics of the transition density function $q(t, x, y)$ of X :

$$\lim_{t \rightarrow 0} t \log q(t, x, y) = -\frac{|x - y|^2}{2} \quad (0.4)$$

for every $x, y \in R^d$.

Part I: A brief history

For non-divergence form elliptic operator $\mathcal{L} = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}$ on R^d with bounded, symmetric and uniformly elliptic diffusion matrix $A(x) = (a_{ij}(x))$ that is uniformly Hölder continuous, Varadhan showed the following small time asymptotics for the heat kernel $p(t, x, y)$ of \mathcal{L} in [V1]

$$\lim_{t \downarrow 0} t \log p(t, x, y) = -d(x, y)^2/2 \quad \text{for } x, y \in R^d, \quad (0.5)$$

where $d(x, y)$ is the Riemannian metric induced by $A(x)^{-1}$. Later, (0.5) is extended by Norris [N] to divergence form elliptic operator

$$\mathcal{L} = \frac{1}{2\rho(x)} \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left(\rho(x) a_{ij}(x) \frac{\partial}{\partial x_j} \right),$$

where $A(x) = (a_{ij}(x))$ is symmetric locally bounded and locally uniformly elliptic and $\rho(x)$ is a measurable function that is locally bounded between two positive constants.

Part I: A brief history

In [AH], Ariyoshi and Hino studied the integral version of (0.5) and showed that for heat semigroup $\{P_t; t \geq 0\}$ associated with any symmetric local Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E; m)$ with σ -finite measure m ,

$$\lim_{t \downarrow 0} t \log P_t(A_1, A_2) = -d(A_1, A_2)^2/2 \quad (0.6)$$

for any Borel subsets $A_i \subset E$ with $0 < m(A_i) < \infty$ for $i = 1, 2$. Here $P_t(A_1, A_2) := \int_{A_2} P_t \mathbf{1}_{A_1}(x) m(dx)$ and $d(A_1, A_2)$ is the intrinsic distance between A_1 and A_2 induced by the local Dirichlet form $(\mathcal{E}, \mathcal{F})$.

Part I: Main results

For $x \in R^d$ and $\varepsilon > 0$, denote by ν_ε^x the distribution of $\{X_{\varepsilon t}; t \in [0, 1]\}$ under P_x on the path space $C([0, 1], R^d)$. We have the following large deviation result.

Theorem 1.1. For each fixed $x \in R^d$, $\{\nu_\varepsilon^x, \varepsilon > 0\}$ satisfies a large deviation principle with rate function

$$I(f) = \begin{cases} \frac{1}{2} \int_0^1 |\dot{f}(t)|^2 dt & \text{if } f \text{ is absolutely continuous,} \\ \infty & \text{otherwise,} \end{cases}$$

where \dot{f} stands for the derivative of $t \rightarrow f(t)$. Namely,

(i) for any closed subset $C \subset C([0, 1], R^d)$,

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon \log \nu_\varepsilon^x(C) \leq - \inf_{f \in C(x)} I(f),$$

where $C(x) = \{f \in C : f(0) = x\}$;

(ii) for any open subset $G \subset C([0, 1], R^d)$,

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon \log \nu_\varepsilon^x(G) \geq - \inf_{f \in G(x)} I(f),$$

where $G(x) = \{f \in G : f(0) = x\}$.

The following result is the precise Varadhan's asymptotics:

Theorem 1.2.

$$\lim_{t \rightarrow 0} t \log q(t, x, y) = -\frac{|x - y|^2}{2} \quad (0.7)$$

uniformly in x, y on compact subsets of $R^d \times R^d$.

Part I: Sketch of the proofs

Recall $X_t = x + W_t + A_t$. To prove the above theorems we first need to establish the following result.

Proposition 1.3. For any $\delta > 0$,

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon \log P_x \left(\sup_{0 \leq t \leq 1} |A_{\varepsilon t}| = \sup_{0 \leq t \leq 1} |X_{\varepsilon t} - (x + W_{\varepsilon t})| > \delta \right) = -\infty. \quad (0.8)$$

Part I: Sketch of the proofs

To prove Proposition 1.3, the following bounds of the Laplace transform of a positive continuous additive functional is crucial.

For a signed measure μ on R^d , we introduce

$$\Lambda_t(\mu) := \sup_{x \in R^d} \int_0^t \int_{R^d} \frac{q(s, x, y)}{\sqrt{s}} |\mu|(dy) ds. \quad (0.9)$$

Proposition 1.4. Let $b(\cdot)$ be a positive function in the Kato class $K_{d,1}$. Then, for any $\lambda > 0$

$$\begin{aligned} & E_x \left[e^{\lambda \int_0^t b(X_s) ds} \right] \\ & \leq \left(1 + \lambda \sqrt{t} \Lambda_t(b) \right) \exp \left(\lambda^2 t \Lambda_t(b)^2 \right) \\ & \leq 2 \exp \left(2 \lambda^2 t \Lambda_t(b)^2 \right), \end{aligned} \tag{0.10}$$

where $\Lambda_t(b)$ is defined as in (0.9).

It is important that the bound on right side only depends on the Kato norm of the measure so that it can be extended to general additive functionals.

Sketch of the proof of Proposition 1.4.

We claim that for non-negative integer $n \geq 0$,

$$E_x \left[\left(\int_0^t b(X_s) ds \right)^n \right] \leq n! \alpha_n \left(\sqrt{t} \Lambda_t(b) \right)^n, \quad (0.11)$$

where $\alpha_0 = \alpha_1 = 1$, $\alpha_n = \prod_{k=2}^n \left(1 - \frac{1}{k}\right)^{(k-1)/2} \left(\frac{1}{k}\right)^{1/2}$ for $n \geq 2$.

Part I: Sketch of the proofs

Indeed, for $n \geq 1$, we have by the Markov property of X ,

$$\begin{aligned} & E_x \left[\left(\int_0^t b(X_s) ds \right)^n \right] \\ &= n! \int_0^t ds_n \int_0^{s_n} ds_{n-1} \cdots \int_0^{s_2} ds_1 E_x [b(X_{s_1}) b(X_{s_2}) \cdots b(X_{s_n})] \\ &= n! \int_{\{0 \leq s_1 < s_2 < \cdots < s_n \leq t\}} ds_1 \cdots ds_n \int_{(R^d)^{\otimes n}} b(y_1) \cdots b(y_n) q(s_1, x, y_1) \\ &\quad \times q(s_2 - s_1, y_1, y_2) \cdots q(s_n - s_{n-1}, y_{n-1}, y_n) dy_n \cdots dy_1. \quad (0.12) \end{aligned}$$

Part I: Sketch of the proofs

When $n = 1$,

$$\begin{aligned} \sup_{x \in \mathbb{R}^d} E_x \left[\int_0^t b(X_s) ds \right] &= \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} b(y) q(s, x, y) dy ds \\ &\leq \sqrt{t} \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} b(y) \frac{q(s, x, y)}{\sqrt{s}} dy \\ &= \sqrt{t} \Lambda_t(b). \end{aligned} \tag{0.13}$$

So (0.11) holds for $n = 0$ and $n = 1$ as claimed. When $n = 2$, by (0.12) and (0.13),

Part I: Sketch of the proofs

$$\begin{aligned} & \sup_{x \in \mathbb{R}^d} E_x \left[\left(\int_0^t b(X_s) ds \right)^2 \right] \\ &= 2 \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} b(y_1) q(s_1, x, y_1) \\ & \quad \left(\int_{s_1}^t \int_{\mathbb{R}^d} b(y_2) q(s_2 - s_1, y_1, y_2) ds_2 dy_2 \right) ds_1 dy_1 \\ &= 2 \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} b(y_1) q(s_1, x, y_1) \\ & \quad \left(\int_0^{t-s_1} \int_{\mathbb{R}^d} b(y_2) q(r, y_1, y_2) dr dy_2 \right) ds_1 dy_1 \\ &\leq 2 \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} b(y_1) q(s_1, x, y_1) \sqrt{t - s_1} \Lambda_t(b) ds_1 dy_1 \end{aligned}$$

Part I: Sketch of the proofs

$$\begin{aligned} &= 2\Lambda_t(b) \sup_{x \in R^d} \int_0^t \int_{R^d} \sqrt{(t-s_1)s_1} b(y_1) \frac{q(s_1, x, y_1)}{\sqrt{s_1}} ds_1 dy_1 \\ &\leq 2! \cdot \frac{1}{2} \cdot \left(\sqrt{t} \Lambda_t(b) \right)^2, \end{aligned}$$

as $\max_{s_1 \in [0, t]} \sqrt{s_1(t-s_1)} = t/2$. This shows that (0.11) holds for $n = 2$.

Part I: Sketch of the proofs

Now assuming (0.11) holds for $n = k \geq 2$, we next show it holds for $n = k + 1$. By (0.11) for $n = k$,

$$\begin{aligned} & \sup_{x \in \mathbb{R}^d} E_x \left[\left(\int_0^t b(X_s) ds \right)^{k+1} \right] \\ & \leq \dots \\ & \leq (k+1)! \alpha_k \Lambda_t(b)^k \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} b(y_1) \frac{q(s_1, x, y_1)}{\sqrt{s_1}} s_1^{1/2} (t - s_1)^{k/2} ds_1 dy_1 \\ & \leq (k+1)! \alpha_k \left(\frac{k}{k+1} \right)^{k/2} \left(\frac{1}{k+1} \right)^{1/2} t^{(k+1)/2} \Lambda_t(b)^{k+1} \\ & = (k+1)! \alpha_{k+1} \left(\sqrt{t} \Lambda_t(b) \right)^{k+1}, \end{aligned}$$

$$\text{as } \max_{s_1 \in [0, s_1]} s_1^{1/2} (t - s_1)^{k/2} = \left(\frac{k}{k+1} \right)^{k/2} \left(\frac{1}{k+1} \right)^{1/2} t^{(k+1)/2}.$$

Part I: Sketch of the proofs

This shows that (0.11) holds for $n = k + 1$. By induction, we have established that (0.11) holds for all $n \geq 1$.

Observe that $\alpha_n \leq \frac{1}{\sqrt{n!}}$. We have by (0.11) that for $\lambda > 0$,

$$\begin{aligned} E_x \left[e^{\lambda \int_0^t b(X_s) ds} \right] &= \sum_{n=0}^{\infty} \frac{1}{n!} E_x \left[\left(\lambda \int_0^t b(X_s) ds \right)^n \right] \\ &\leq \sum_{n=0}^{\infty} \frac{(\lambda \sqrt{t} \Lambda_t(b))^n}{\sqrt{n!}}. \end{aligned} \quad (0.14)$$

Since $(2n+1)! \geq (2n)! \geq (n!)^2$ for integer $n \geq 0$, we have for $z \geq 0$,

$$\begin{aligned} \Phi(z) &:= \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} = \sum_{n=0}^{\infty} \frac{z^{(2n)}}{\sqrt{(2n)!}} + \sum_{n=0}^{\infty} \frac{z^{2n+1}}{\sqrt{(2n+1)!}} \\ &\leq \sum_{n=0}^{\infty} \frac{z^{2n}}{n!} + z \sum_{n=0}^{\infty} \frac{z^{2n}}{n!} = (1+z)e^{z^2}. \end{aligned} \quad (0.15)$$

This combined with (0.14) yields that

$$E_x \left[e^{\lambda \int_0^t b(X_s) ds} \right] \leq \left(1 + \lambda \Lambda_t(b) \sqrt{t} \right) e^{\lambda^2 \Lambda_t(b)^2 t}. \quad (0.16)$$

This completes the proof of the proposition as $1 + a \leq 2e^{a^2}$ for every $a \geq 0$. □

Sketch of the proof of Theorem 1.2.

Step 1. Upper bound. We will prove

$$\limsup_{t \rightarrow 0} t \log q(t, x, y) \leq -\frac{|x - y|^2}{2}$$

uniformly in $x, y \in R^d$ such that $|x - y|$ is bounded.

Part I: Sketch of the proofs

It is sufficient to show that for any $\delta > 0$, there exist constants $T_\delta > 0$, $C_{2,\delta}$ such that for $t \in (0, T_\delta]$,

$$q(t, x, y) \leq C_{2,\delta} t^{-d/2} \exp\left(- (1 - \delta) \frac{|x - y|^2}{2t}\right). \quad (0.17)$$

Part I: Sketch of the proofs

Define recursively $l_k(t, x, y)$ as follows:

$$l_0(t, x, y) = p(t, x, y) = (2\pi t)^{-d/2} \exp\left(-\frac{|x - y|^2}{2t}\right),$$

$$l_{k+1}(t, x, y) = \int_0^t \int_{R^d} l_k(t - s, x, z) b(z) \cdot \nabla_z p(s, z, y) dz ds \quad \text{for } k \geq 0$$

We have the following expansion (see, [KS]):

$$q(t, x, y) = \sum_{k=0}^{\infty} l_k(t, x, y). \quad (0.18)$$

Part I: Sketch of the proofs

For a signed measure μ on R^d , define

$$N_t^\alpha(\mu) := \sup_{x \in R^d} \int_0^t \int_{R^d} s^{-(d+1)/2} \exp\left(-\alpha \frac{|x-y|^2}{s}\right) |\mu|(dy) ds,$$

where $|\mu|$ denotes the measure of total variation.

Part I: Sketch of the proofs

We next show that for any $\delta \in (0, 1)$, it holds that

$$|I_k(t, x, y)| \leq (C_\delta N_t^{c_\delta}(|b|))^k t^{-d/2} \exp\left(- (1 - \delta) \frac{|x - y|^2}{2t}\right). \quad (0.19)$$

(0.19) clearly holds for $k = 0, 1$. Suppose (0.19) holds for $k \geq 1$. By induction, we have

$$\begin{aligned} I_{k+1}(t, x, y) &\leq (C_\delta N_t^{\delta/2}(|b|))^k \int_0^t \int_{R^d} G_{1-\delta}(t-s, x, z) \\ &\quad \times |b(z)| |\nabla_z G_1(s, z, y)| dz ds \\ &\leq (C_\delta N_t^{c_\delta}(|b|))^k C_\delta N_t^{c_\delta}(|b|) G_{1-\delta}(t, x, y) \\ &= (C_\delta N_t^{c_\delta}(|b|))^{k+1} t^{-d/2} \exp\left(- (1 - \delta) \frac{|x - y|^2}{2t}\right). \end{aligned}$$

where $G_a(s, x, y) = s^{-d/2} \exp\left(-\frac{a|x-y|^2}{2s}\right)$.

Part I: Sketch of the proofs

Since $\mu = b(x)dx \in K_{d,1}$, there exists a constant $T_\delta > 0$ such that $C_\delta N_t^{\delta/2}(|b|) \leq \frac{1}{2}$ for $t \leq T_\delta$. This together with (0.18), (0.19) gives that

$$q(t, x, y) \leq 2t^{-d/2} \exp\left(- (1 - \delta) \frac{|x - y|^2}{2t}\right).$$

From the proof above, we see that the constant T_δ only depends on the rate at which $N_t^{\delta/2}(|b|)$ tends to zero. Consequently,

$$t \log q(t, x, y) \leq - (1 - \delta) \frac{|x - y|^2}{2} - (d/2)t \log(2t) \quad \text{for } t \in (0, T_\delta].$$

It follows that $\limsup_{t \rightarrow 0} t \log q(t, x, y) \leq -\frac{|x-y|^2}{2}$ uniformly on compact subsets of $R^d \times R^d$. By an approximation procedure as in [KS], we assert that the Proposition hold also for $\mu \in K_{d,1}$.

Step 2. Lower bound. We need to prove

$$\liminf_{t \rightarrow 0} t \log q(t, x, y) \geq -\frac{|x - y|^2}{2} \quad (0.20)$$

uniformly in $x, y \in R^d$ such that $|x - y|$ is bounded.

Part I: Sketch of the proofs

For $\varepsilon > 0$, set $B(y, \varepsilon) = \{z; |z - y| < \varepsilon\}$. Let $r > 0$. We first like to give an estimate for the probability $P_x(X_r \in B(y, \varepsilon))$. Let $\delta > 0$ be any constant such that $\delta < \varepsilon$. We have

$$\begin{aligned} P_x(W_r + x \in B(y, \varepsilon - \delta)) &= P_x(X_r - A_r \in B(y, \varepsilon - \delta)) \\ &\leq P_x(X_r - A_r \in B(y, \varepsilon - \delta), |A_r| < \delta) + P_x(|A_r| \geq \delta) \\ &\leq P_x(X_r \in B(y, \varepsilon)) + P_x(|A_r| \geq \delta). \end{aligned} \quad (0.21)$$

By Proposition 4,

$$\begin{aligned} &P_x(|A_r| \geq \delta) \\ &\leq 2 \exp \left(- \frac{\delta^2}{8rC_4^2 e^{2C_5} N_r^{C_6} \left(\sum_{i=1}^d |\mu_i| \right)^2} \right). \end{aligned} \quad (0.22)$$

Part I: Sketch of the proofs

On the other hand,

$$\begin{aligned} P_x(W_r + x \in B(y, \varepsilon - \delta)) &= \int_{B(0, \varepsilon - \delta)} (2\pi r)^{-\frac{d}{2}} e^{-\frac{|x-y-z|^2}{2r}} dz \\ &\geq c(2\pi r)^{\frac{d}{2}} \exp\left(-\frac{(|x-y| + \varepsilon - \delta)^2}{2r}\right) (\varepsilon - \delta)^d. \end{aligned} \quad (0.23)$$

Putting together (0.22) and (0.23), we obtain for $\delta < \varepsilon$,

Part I: Sketch of the proofs

$$\begin{aligned} & P_x(X_r \in B(y, \varepsilon)) \\ & \geq c(2\pi r)^{-\frac{d}{2}}(\varepsilon - \delta)^d \exp\left(-\frac{(|x - y| + \varepsilon - \delta)^2}{2r}\right) \\ & \quad \times \left[1 - 2c^{-1}(2\pi r)^{\frac{d}{2}}(\varepsilon - \delta)^{-d} \exp\left(-\left[\frac{\delta^2}{8rC_4^2 e^{2C_5} N_r^{C_6} \left(\sum_{i=1}^d |\mu_i|\right)^2} - \frac{(|x - y| + \varepsilon - \delta)^2}{2r}\right]\right)\right] \end{aligned} \tag{0.24}$$

Part I: Sketch of the proofs

Now for every η with $0 < \eta < 1$ and $\varepsilon > 0$ we have

$$\begin{aligned} q(t, x, y) &= \int_{R^d} q((1-\eta)t, x, z) q(\eta t, z, y) dz \\ &\geq \int_{B(y, \varepsilon)} q((1-\eta)t, x, z) q(\eta t, z, y) dz \\ &\geq \inf_{z \in B(y, \varepsilon)} q(\eta t, z, y) \int_{B(y, \varepsilon)} q((1-\eta)t, x, z) dz \\ &= \inf_{z \in B(y, \varepsilon)} q(\eta t, z, y) P_x (X_{(1-\eta)t} \in B(y, \varepsilon)). \quad (0.25) \end{aligned}$$

By (0.3),

$$\inf_{z \in B(y, \varepsilon)} q(\eta t, z, y) \geq C_1 e^{-C_2 t} (\eta t)^{-d/2} \exp\left(-C_3 \frac{\varepsilon^2}{2\eta t}\right). \quad (0.26)$$

Part I: Sketch of the proofs

Using (0.24) with $r = (1 - \eta)t$, it follows from (0.25), (0.26) that for $\delta < \varepsilon$,

$$\begin{aligned} & q(t, x, y) \\ & \geq C_1 e^{-C_2 t} (\eta t)^{-d/2} \exp\left(-C_3 \frac{\varepsilon^2}{2\eta t}\right) c(2\pi(1 - \eta)t)^{-\frac{d}{2}} \\ & \quad \times (\varepsilon - \delta)^d \exp\left(-\frac{(|x - y| + \varepsilon - \delta)^2}{2(1 - \eta)t}\right) \end{aligned}$$

$$\begin{aligned}
 & \times \left[1 - 2c^{-1}(2\pi(1-\eta)t)^{\frac{d}{2}}(\varepsilon - \delta)^{-d} \exp \left(\right. \right. \\
 & \quad \left. \left. - \left[\frac{\delta^2}{8(1-\eta)tC_4^2 e^{2C_5} N_{(1-\eta)t}^{C_6} \left(\sum_{i=1}^d |\mu_i| \right)^2} \right. \right. \right. \\
 & \quad \left. \left. \left. - \frac{(|x-y| + \varepsilon - \delta)^2}{2(1-\eta)t} \right] \right) \right] \tag{0.27}
 \end{aligned}$$

Part I: Sketch of the proofs

If $|x - y| \leq M$ for some M , then (0.27) yields that

$$\begin{aligned} & t \log q(t, x, y) \\ \geq & t \log \left(C_1 e^{-C_2 t} (\eta t)^{-d/2} c(2\pi(1-\eta)t)^{-\frac{d}{2}} (\varepsilon - \delta)^d \right) \\ & - C_3 \frac{\varepsilon^2}{2\eta} - \frac{(|x - y| + \varepsilon - \delta)^2}{2(1-\eta)} \\ & + t \log \left(1 - 2c^{-1} (2\pi(1-\eta)t)^{\frac{d}{2}} (\varepsilon - \delta)^{-d} \right. \\ & \left. \exp \left(- \left[\frac{\delta^2}{8(1-\eta)t C_4^2 e^{2C_5} N_{(1-\eta)t}^{C_6} \left(\sum_{i=1}^d |\mu_i| \right)^2} \right. \right. \right. \\ & \left. \left. \left. - \frac{(M + \varepsilon - \delta)^2}{2(1-\eta)t} \right] \right) \right) \end{aligned} \tag{0.28}$$

Part I: Sketch of the proofs

Note that $N_{(1-\eta)t}^{C_6} \left(\sum_{i=1}^d |\mu_i| \right)^2 \rightarrow 0$ as $t \rightarrow 0$. Let $t \rightarrow 0$ in (0.29) to get

$$\liminf_{t \rightarrow 0} t \log q(t, x, y) \geq -C_3 \frac{\varepsilon^2}{2\eta} - \frac{(|x - y| + \varepsilon - \delta)^2}{2(1 - \eta)} \quad (0.29)$$

uniformly in x, y with $|x - y| \leq M$. Letting first $\varepsilon \rightarrow 0$ and then $\eta \rightarrow 0$ we arrive that

$$\liminf_{t \rightarrow 0} t \log q(t, x, y) \geq -\frac{|x - y|^2}{2}.$$

The proof is complete. □

Part II: Formulation of the problem

Let D be a bounded, connected $C^{1,1}$ -domain and $\varphi \in C(\partial D)$. Consider the following Dirichlet boundary value problem:

$$\begin{cases} \mathcal{A}u = -\rho, & \forall x \in D, \\ u(x)|_{\partial D} = \varphi(x), & \forall x \in \partial D, \end{cases} \quad (0.30)$$

where

$$\mathcal{A}u := \frac{1}{2}\Delta u + \nabla u \cdot \mu + u\nu, \quad (0.31)$$

and the signed measures μ, ν, ρ Kato class measures. Our aim is to study the well posedness of the boundary value problem (0.30). Here is the precise meaning of a solution.

Part II: Formulation of the problem

For $u, v \in C_0^\infty(D)$, define the quadratic form

$$\begin{aligned} \mathcal{Q}(u, v) &:= \frac{1}{2} \sum_{i=1}^d \int_D \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_i} dx - \sum_{i=1}^d \int_D \frac{\partial u(x)}{\partial x_i} v(x) \mu_i(dx) \\ &\quad - \int_D u(x) v(x) \nu(dx). \end{aligned} \tag{0.32}$$

Part II: Formulation of the problem

Definition 2.1. A function $u \in C^1(D) \cap C(\overline{D})$ is called a weak solution to the boundary value problem

$$\begin{cases} \mathcal{A}u = -\rho, & \forall x \in D, \\ u(x)|_{\partial D} = \varphi(x), & \forall x \in \partial D, \end{cases} \quad (0.33)$$

if u satisfies

$$\mathcal{Q}(u, \phi) = \int_D \phi(x) \rho(dx) \quad \text{for any } \phi \in C_0^\infty(D),$$

and $u(x) = \varphi(x)$ when $x \in \partial D$.

Remark. Since the signed measures involved are in general not absolutely continuous *w.r.t.* dx , the classical notion of weak solution in the Sobolev space $W^{1,2}(D)$ is not suitable here.

Part II: Several words about the history

As far as we know, this is the first paper (even in the PDE literature) to study the boundary value problems for elliptic operators with drifts being signed measures. Our approach is probabilistic, which not only provides existence and uniqueness of the solution, and also gives the solution a representation. Using probabilistic approaches to solve boundary value problems has a long history. Many interesting work have been done. We mention a few names: S. Kakutani, K.L.Chung, Z. Zhao, Z.M. Ma, R. Song, Z.Q. Chen, Z.Q.Chen and Z. , W.D. Gerhard etc.

Part II: The associated diffusion process

Recall that $\mathcal{L}u := \frac{1}{2}\Delta u + \nabla u \cdot \mu$ is the infinitesimal generator of the diffusion process

$$X_t = X_0 + W_t + A_t,$$

For $n \geq 1$, let X_t^n denote the solution of the following stochastic differential equation:

$$X_t^n = x + W_t + \int_0^t b^{(n)}(X_s^n) ds, \quad P_x - a.e.. \quad (0.34)$$

Denote by $q^D(t, x, y)$ the transition density function of the killed process X^D of X upon leaving the domain D . $q^n(t, x, y)$ and $q^{n,D}(t, x, y)$ denote the transition density functions of the diffusion process X^n and the killed process $X^{n,D}$.

Part II: Additive functionals

Definition 2.2. Fix a constant $c > C_5$. For any signed measure π in \mathbb{R}^d , we say that a continuous additive functional (CAF) B_t of X is associated with π , if

$$B_t = B_t^+ - B_t^-, \quad \forall t > 0,$$

where B_t^+ and B_t^- are positive CAFs satisfy that for any $x \in \mathbb{R}^d$,

$$\begin{aligned} E_x \left[\int_0^\infty e^{-ct} dB_t^+ \right] &= \int_0^\infty e^{-ct} \int_{\mathbb{R}^d} q(t, x, y) \pi^+(dy) dt, \\ E_x \left[\int_0^\infty e^{-ct} dB_t^- \right] &= \int_0^\infty e^{-ct} \int_{\mathbb{R}^d} q(t, x, y) \pi^-(dy) dt. \end{aligned} \quad (0.35)$$

Part II: Additive functionals

For a signed measure π in \mathbb{R}^d , set

$$H_n(x) := \int_{\mathbb{R}^d} \psi_n(x-y)\pi(dy). \quad (0.36)$$

Regarding the existence and uniqueness of CAF, we have the following result.

Proposition 2.1. Fix a constant $c > C_5$. Then for any signed measure $\pi \in \mathbf{K}_{d,2}$ with compact support, there exists a unique CAF $B_t = B_t^+ - B_t^-$ associated with the measure π in the sense that, B_t^+ and B_t^- are positive CAFs such that for any $x \in \mathbb{R}^d$,

$$E_x\left[\int_0^\infty e^{-ct} dB_t^+\right] = \int_0^\infty e^{-ct} \int_{\mathbb{R}^d} q(t, x, y)\pi^+(dy)dt, \quad (0.37)$$

$$E_x\left[\int_0^\infty e^{-ct} dB_t^-\right] = \int_0^\infty e^{-ct} \int_{\mathbb{R}^d} q(t, x, y)\pi^-(dy)dt. \quad (0.38)$$

Part II: Additive functionals

Moreover, for any $x \in \mathbb{R}^d$ and constant $T > 0$,

$$\lim_{n \rightarrow \infty} \sup_{t \leq T} |B_t - \int_0^t H_n(X_s) ds| = 0, \text{ in } P_x, \quad (0.39)$$

and if $x \in D$, then B_{τ_D} is integrable *w.r.t.* P_x and

$$E_x[B_{\tau_D}] = \int_0^\infty \int_D q^D(t, x, y) \pi(dy) dt. \quad (0.40)$$

Let $\varphi \in C(\partial D)$ and consider the following boundary problem:

$$\begin{cases} \frac{1}{2}\Delta u + \nabla u \cdot \mu = -\rho, & \forall x \in D, \\ u(x)|_{\partial D} = \varphi(x), & \forall x \in \partial D. \end{cases} \quad (0.41)$$

Let $V_t, t \geq 0$ be the continuous additive functional associated with the measure ρ . Here is the main result on the existence of solution.

Theorem 2.1. Assume $\varphi \in C^{1,\alpha_0}(\partial D)$ and the boundary ∂D is $C^{1,1}$. Let $u(x) := E_x[\varphi(X_{\tau_D}) + V_{\tau_D}]$. Then $u \in C_b^1(D)$ is a weak solution to Dirichlet boundary problem (0.41).

Moreover, if for each $1 \leq i \leq d$, $\mu_i, \rho \in \mathbf{K}_{d,1-\alpha_0}$, then $u \in C^{1,\alpha_0}(\bar{D})$ and there exists a $c > 0$, depending on μ only via the function $\max_{1 \leq i \leq d} M_{\mu_i}^{1-\alpha_0}(\cdot)$, such that

$$\|u\|_{C^{1,\alpha_0}(\bar{D})} \leq c(\|\varphi\|_{C^{1,\alpha_0}(\partial D)} + M_{\rho}^{1-\alpha_0}(R_0)). \quad (0.42)$$

The proof of Theorem 2.1 has three main ingredients:

(I). Prove the Hölder continuity of the heat kernel of the killed process.

Proposition 2.2 Assume for each $1 \leq i \leq d$, $\mu_i \in \mathbf{K}_{d,1-\alpha_0}$. Then for $t > 0, y \in D$, $q^D(t, \cdot, y) \in C^{1,\alpha_0}(\overline{D})$. Moreover there exist constants $C_{14}, C_{15} > 0$, depending on μ only via the function $\max_{1 \leq i \leq d} M_{\mu_i}^{1-\alpha_0}(\cdot)$, such that for any convex subset $D' \subset D$, $(t, x, y), (t, x', y) \in (0, \infty) \times \overline{D'} \times D$ and $1 \leq j \leq d$,

$$\begin{aligned} |\partial_{x_j} q^D(t, x, y) - \partial_{x_j} q^D(t, x', y)| &\leq C_{14} |x - x'|^{\alpha_0} t^{-\frac{d+1+\alpha_0}{2}} \\ &\times \left(e^{-\frac{C_{15}|x-y|^2}{2t}} + e^{-\frac{C_{15}|x'-y|^2}{2t}} \right). \end{aligned} \quad (0.43)$$

(II). Prove that for any $x_0 \in \partial D$,

$$\lim_{D \ni x \rightarrow x_0} u(x) = \varphi(x_0).$$

(III). Approximation

Set $K_n(x) := \int_{\mathbb{R}^d} \psi_n(x-y)\rho(dy)$ and

$$u_n(x) := E_x[\varphi(X_{\tau_D^n}^n)] + E_x\left[\int_0^{\tau_D^n} K_n(X_s^n) ds\right].$$

Proposition 2.3. For any compact subset $K \subset D$,

$$\lim_{n \rightarrow \infty} \sup_{x \in K} |u_n - u(x)| = 0, \quad (0.44)$$

and

$$\lim_{n \rightarrow \infty} \sup_{x \in K} |\nabla u_n - \nabla u(x)| = 0. \quad (0.45)$$

The uniqueness of solution is given in the next theorem.

Theorem 2.2. Assume $\mu, \rho \in \mathbf{K}_{d,1-\alpha_0}$ and $\varphi \in C(\partial D)$. Then there exists a unique weak solution u to problem (0.41).

We first prove the uniqueness when the domain D is a small ball and then complete the proof by covering the given domain D by countably many small balls.

We now consider the following general problem:

$$\begin{cases} \frac{1}{2}\Delta u + \nabla u \cdot \mu + u\nu = -\rho, & \forall x \in D, \\ u(x)|_{\partial D} = \varphi(x), & \forall x \in \partial D. \end{cases} \quad (0.46)$$

Let L_t denote the CAF associated with the signed measure ν . Recall that V_t is the CAF associated with ρ . Here is the result.





Theorem 2.3. Let $\varphi \in C^{1,\alpha_0}(\partial D)$. Assume there exists a $x_0 \in D$ such that

$$E_{x_0}[e^{L\tau_D}] < \infty.$$





Then




$$u(x) := E_x[e^{L\tau_D} \varphi(X_{\tau_D}) + \int_0^{\tau_D} e^{Ls} dV_s]. \quad (0.47)$$



is a weak solution to the Dirichlet boundary value problem (0.46). Moreover, if for each $1 \leq i \leq d$, μ_i , ν and ρ belong to $\mathbf{K}_{d,1-\alpha_0}$, then u defined above is the unique solution.

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